SCHRÖDINGER EQUATIONS ON DAMEK-RICCI SPACES

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ABSTRACT. In this paper we consider the Laplace-Beltrami operator Δ on Damek-Ricci spaces and derive pointwise estimates for the kernel of $e^{\tau\Delta}$, when $\tau \in \mathbb{C}^*$ with $\operatorname{Re}\tau \geq 0$. When $\tau \in i\mathbb{R}^*$, we obtain in particular pointwise estimates of the Schrödinger kernel associated with Δ. We then prove Strichartz estimates for the Schrödinger equation, for a family of admissible pairs which is larger than in the Euclidean case. This extends the results obtained by Anker and Pierfelice [4] on real hyperbolic spaces. As a further application, we study the dispersive properties of the Schrödinger equation associated with a distinguished Laplacian on Damek-Ricci spaces, showing that in this case the standard $L^1 \to L^\infty$ estimate fails while suitable weighted Strichartz estimates hold.

1. Introduction

The study of the dispersive properties of many evolution equations of mathematical physics, including the Schrödinger and heat equation on $\mathbb{R}^n$, is of fundamental importance. Indeed, dispersive estimates represent the main tool in the study of several linear and nonlinear problems. We recall some standard facts. Consider the homogeneous Cauchy problem for the linear Schrödinger equation on $\mathbb{R}^n$, $n \geq 1$,

$$\begin{cases} i\,\partial_t u(t,x) + \Delta u(t,x) = 0 \\ u(0,x) = f(x), \end{cases} \tag{1}$$

whose solution can be represented as

$$u(t,x) = e^{it\Delta} f(x) = \frac{1}{(4\pi i t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\frac{|x-y|^2}{4t}} f(y)\,dy \qquad \forall t \neq 0\,.$$

By the explicit representation of the kernel of $e^{it\Delta}$ one easily obtains the dispersive estimate

$$\|e^{it\Delta}\|_{L^1(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n)} \lesssim |t|^{-\frac{n}{2}} \qquad \forall t \neq 0\,.$$

It is sufficient to get rid of i in the kernel to obtain a corresponding representation for the heat kernel of $e^{t\Delta}$, and hence a similar dispersive estimate

$$\|e^{t\Delta}\|_{L^1(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n)} \lesssim t^{-\frac{n}{2}} \qquad \forall t > 0\,.$$

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It is well known that, starting from the dispersive estimates, it is possible to deduce other space-time estimates which are called Strichartz estimates. The first such estimate was obtained by Strichartz himself in a special case; then Ginibre and Velo [27] obtained the complete range of estimates with the exclusion of some critical cases, the *endpoint* cases, which were finally proved by Keel and Tao [34]. We recall that the modern theory of local and global well posedness for semilinear Schrödinger equations is based essentially on these estimates.

In view of the important applications to nonlinear problems, many attempts have been made to study the dispersive properties for the corresponding equations on various Riemannian manifolds (see e.g. [4, 6, 7, 10, 11, 28, 32, 40, 41] among the others).

More precisely, dispersive and Strichartz estimates for the Schrödinger equation on real hyperbolic spaces $\mathbb{H}^n$, which are manifolds with constant negative curvature, have been stated by Banica, Anker and Pierfelice, Ionescu and Staffilani ([4, 6, 7, 32, 40, 41]). Here we are interested in extending these results to the more general context of *Damek-Ricci spaces*, also known as *harmonic NA groups* ([2, 9, 15, 16, 22, 23, 24, 25, 42]). As Riemannian manifolds, these solvable Lie groups include all symmetric spaces of noncompact type and rank one, but most of them are not symmetric, thus providing counterexemples to the Lichnerowicz conjecture [24].

We briefly recall the definition of the spaces. Let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ be an Heisenberg-type algebra and let N be the connected and simply connected Lie group associated to $\mathfrak{n}$ (see Section 2 for the details). Let S be the one-dimensional extension of N obtained by making $A = \mathbb{R}^+$ act on N by homogeneous dilations. We denote by Q the homogeneous dimension of N and by n the dimension of S. Let H denote a vector in $\mathfrak{a}$ acting on $\mathfrak{n}$ with eigenvalues $1/2$ and (possibly) 1; we extend the inner product on $\mathfrak{n}$ to the algebra $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$, by requiring $\mathfrak{n}$ and $\mathfrak{a}$ to be orthogonal and H to be a unit vector. We denote by d the left invariant distance on S associated with the Riemannian metric on S which agrees with the inner product on $\mathfrak{s}$ at the identity. The Riemannian manifold (S, d) is usually referred to as *Damek-Ricci space.*

Note that S is nonunimodular in general; denote by λ and ρ the left and right Haar measures on S, respectively. It is well known that the spaces (S, d, λ) and (S, d, ρ) are of *exponential growth.* In particular, the two following Laplacians on S have been the object of investigation :

(i) The Laplace-Beltrami operator Δ_S associated with the Riemannian metric d. The operator $-\Delta_S$ is left invariant, it is essentially selfadjoint on $L^2(S, \lambda)$ and its spectrum is the half line $[Q^2/4, \infty)$.

(ii) The left invariant Laplacian $\mathcal{L} = \sum_{i=0}^{n-1} X_i^2$, where $X_0, ...X_{n-1}$ are left invariant vector fields such that at the identity $X_0 = H$, $\{X_1, \ldots, X_{m_\mathfrak{v}}\}$ is an orthonormal basis of $\mathfrak{v}$ and $\{X_{m_\mathfrak{v}+1}, \ldots, X_{n-1}\}$ is an orthonormal basis of $\mathfrak{z}$. The operator $-\mathcal{L}$ is essentially selfadjoint on $L^2(S, \rho)$ and its spectrum is $[0, \infty)$.

Considerable effort has been produced to study the so-called L^p–functional calculus for the operators $-\Delta_S$ and $-\mathcal{L}$. It turned out that if $p \neq 2$, then $-\Delta_S$ possesses a L^p *holomorphic* functional calculus [12], whereas $-\mathcal{L}$ admits a L^p functional calculus *of Mihlin-Hörmander type* [5, 18]. This interesting dichotomy between the two operators

motivated many authors to study both of them in the context of real hyperbolic spaces, in noncompact symmetric spaces of rank one or, more generally, in Damek-Ricci spaces and in noncompact symmetric spaces of arbitrary rank [1, 3, 5, 19, 20, 21, 29, 30, 31, 38, 39, 43].

In this paper we study the dispersive properties of the Schrödinger equations on S associated with both the Laplacians Δ_S and $\mathcal{L}$.

To this end, in Section 3 we start by proving pointwise estimates of the kernel of the more general operator $e^{\tau\Delta_S}$, for $\tau \in \mathbb{C}^*$ with $\operatorname{Re}\tau \geq 0$. These can be thought as estimates of the heat kernel of the Laplacian Δ_S in complex time and are obtained using the inversion formula for the Abel transform. Similar results were proved in [26, 37] on real hyperbolic spaces.

In the special case when $\operatorname{Re}\tau = 0$ this gives pointwise estimates of the Schrödinger kernel of $e^{it\Delta_S}$, for $t \in \mathbb{R}^*$. These imply the following *dispersive estimates:*

$$\|e^{it\Delta_S}\|_{L^{\tilde{q}'}(S,\lambda)\to L^q(S,\lambda)} \lesssim \begin{cases} |t|^{-\max\{\frac{1}{2}-\frac{1}{q},\frac{1}{2}-\frac{1}{\tilde{q}}\}n} & \text{if } 0<|t|<1 \\ |t|^{-\frac{3}{2}} & \text{if } |t| \geq 1\,, \end{cases}$$

for all $q, \tilde{q} \in (2, \infty]$. As a consequence, we deduce that the solution u of the nonhomogeneous Cauchy problem

$$\begin{cases} i\,\partial_t u(t,x) + \Delta_S u(t,x) = F(t,x) \\ u(0,x) = f(x)\,, \quad x \in S, \end{cases} \tag{2}$$

satisfies the following *Strichartz estimates*

$$\|u\|_{L^p(\mathbb{R};\,L^q(S,\lambda))} \lesssim \|f\|_{L^2(S,\lambda)} + \|F\|_{L^{\tilde{p}'}(\mathbb{R};\,L^{\tilde{q}'}(S,\lambda))}\,, \tag{3}$$

for all couples $(\frac{1}{p}, \frac{1}{q})$ and $(\frac{1}{\tilde{p}}, \frac{1}{\tilde{q}})$ which lie in the *admissible triangle*

$$T_n = \Big\{\Big(\frac{1}{p},\frac{1}{q}\Big) \in \Big(0,\frac{1}{2}\Big] \times \Big(0,\frac{1}{2}\Big) :\ \frac{2}{p}+\frac{n}{q} \geq \frac{n}{2}\Big\} \cup \Big\{\Big(0,\frac{1}{2}\Big)\Big\}\,.$$

Note that the set T_n of admissible pairs for S is much wider that the admissible interval I_n for $\mathbb{R}^n$ which is just the lower edge of the triangle. This phenomenon was already observed by Anker and Pierfelice for real hyperbolic spaces [4] and here is generalized to Damek-Ricci spaces.

As an application of the estimates (3), we study the dispersive properties of the *Schrödinger equation associated with* $\mathcal{L}$:

$$\begin{cases} i\,\partial_t u(t,x) + \mathcal{L}u(t,x) = F(t,x) \\ u(0,x) = f(x)\,, \quad x \in S\,. \end{cases} \tag{4}$$

In this case we prove that there is no dispersive $L^1 - L^\infty$ estimate for the solution of the homogeneous Cauchy problem. Beside this, we are able to show that the solution of the nonhomogeneous Cauchy problem (4) satisfies suitable weighted Strichartz estimates for couples $(\frac{1}{p}, \frac{1}{q})$ and $(\frac{1}{\tilde{p}}, \frac{1}{\tilde{q}})$ in the admissible triangle T_n. More precisely, we obtain this result as an application of the Strichartz estimates proved for the equation associated with Δ_S using a special relationship between the two Laplacians (see (10)).

Note that, in the particular case of real hyperbolic spaces, D. Müller and C. Thiele found a similar lack of the dispersive effect for the wave equation associated with $\mathcal{L}$ and they suggested that Strichartz estimates shall not hold in that case [38, Remark 7.2].

2. Damek-Ricci spaces

In this section we recall the definition of H-type groups, describe their Damek-Ricci extensions, and recall the main results of spherical analysis on these spaces. For the details see [2, 5, 15, 16, 22, 23, 24, 25].

Let $\mathfrak{n}$ be a Lie algebra equipped with an inner product $\langle\cdot,\cdot\rangle$ and denote by $|\cdot|$ the corresponding norm. Let $\mathfrak{v}$ and $\mathfrak{z}$ be complementary orthogonal subspaces of $\mathfrak{n}$ such that $[\mathfrak{n},\mathfrak{z}]=\{0\}$ and $[\mathfrak{n},\mathfrak{n}]\subseteq\mathfrak{z}$. According to Kaplan [33], the algebra $\mathfrak{n}$ is of H-type if for every Z in $\mathfrak{z}$ of unit length the map $J_Z:\mathfrak{v}\to\mathfrak{v}$, defined by

$$\langle J_Z X, Y\rangle = \langle Z,[X,Y]\rangle \qquad \forall X,Y\in\mathfrak{v}\,,$$

is orthogonal. The connected and simply connected Lie group N associated to $\mathfrak{n}$ is called an H-type group. We identify N with its Lie algebra $\mathfrak{n}$ via the exponential map

$$\begin{array}{rcl}\mathfrak{v}\times\mathfrak{z} & \to & N\\ (X,Z) & \mapsto & \exp(X+Z)\,.\end{array}$$

The product law in N is

$$(X,Z)(X',Z') = \Big(X+X', Z+Z'+\frac{1}{2}\,[X,X']\Big) \qquad \forall X,\,X'\in\mathfrak{v}\quad \forall Z,\,Z'\in\mathfrak{z}\,.$$

The group N is a two-step nilpotent group, hence unimodular, with Haar measure $\mathrm{d}X\,\mathrm{d}Z$. We define the following dilations on N:

$$\delta_a(X,Z) = (a^{1/2}X, aZ) \qquad \forall (X,Z)\in N \quad \forall a\in\mathbb{R}^+\,.$$

Set $Q=(m+2k)/2$, where m and k denote the dimensions of $\mathfrak{v}$ and $\mathfrak{z}$, respectively. The number Q is called the homogeneous dimension of N.

Let S be the one-dimensional extension of N obtained by making $A=\mathbb{R}^+$ act on N by homogeneous dilations. We shall denote by n the dimension $m+k+1$ of S. Let H denote a vector in $\mathfrak{a}$ acting on $\mathfrak{n}$ with eigenvalues $1/2$ and (possibly) 1; we extend the inner product on $\mathfrak{n}$ to the algebra $\mathfrak{s}=\mathfrak{n}\oplus\mathfrak{a}$, by requiring $\mathfrak{n}$ and $\mathfrak{a}$ to be orthogonal and H to be a unit vector. The map

$$\begin{array}{rcl}\mathfrak{v}\times\mathfrak{z}\times\mathbb{R}^+ & \to & S\\ (X,Z,a) & \mapsto & \exp(X+Z)\exp(\log a\, H)\end{array}$$

gives global coordinates on S. The product in S is given by the rule

$$(X,Z,a)(X',Z',a') = \Big(X+a^{1/2}X', Z+a\,Z'+\frac{1}{2}\,a^{1/2}[X,X'], a\,a'\Big)$$

for all (X,Z,a), $(X',Z',a')\in S$. The group S is nonunimodular: the right and left Haar measures on S are given by

$$\mathrm{d}\rho(X,Z,a) = a^{-1}\,\mathrm{d}X\,\mathrm{d}Z\,\mathrm{d}a \qquad \text{and} \qquad \mathrm{d}\lambda(X,Z,a) = a^{-(Q+1)}\,\mathrm{d}X\,\mathrm{d}Z\,\mathrm{d}a\,.$$

Then the modular function is $\delta(X,Z,a) = a^{-Q}$. For $p \in [1,\infty)$ we denote by $L^p(S,\lambda)$ and $L^p(S,\rho)$ the spaces of all measurable functions f such that $\int_S |f|^p \,\mathrm{d}\lambda < \infty$ and $\int_S |f|^p \,\mathrm{d}\rho < \infty$, respectively.

We equip S with the left invariant Riemannian metric which agrees with the inner product on $\mathfrak{s}$ at the identity. From [2, formula (2.18)], for all (X,Z,a) in S,

$$\cosh^2\left(\frac{r(X,Z,a)}{2}\right) = \left(\frac{a^{1/2}+a^{-1/2}}{2} + \frac{1}{8}\,a^{-1/2}|X|^2\right)^2 + \frac{1}{4}\,a^{-1}|Z|^2\,, \tag{5}$$

where $r(X,Z,a)$ denotes the distance of the point (X,Z,a) from the identity.

We denote by Δ_S the Laplace-Beltrami operator associated with this Remannian structure on S.

A radial function on S is a function that depends only on the distance from the identity. If f is radial, then by [2, formula (1.16)],

$$\int_S f \,\mathrm{d}\lambda = \int_0^\infty f(r)\,A(r)\,\mathrm{d}r\,,$$

where

$$A(r) = 2^{m+k}\sinh^{m+k}\left(\frac{r}{2}\right)\cosh^k\left(\frac{r}{2}\right) \qquad \forall r \in \mathbb{R}^+\,. \tag{6}$$

A radial function ϕ is spherical if it is an eigenfunction of Δ_S and $\phi(e) = 1$. For s in $\mathbb{C}$, let ϕ_s be the spherical function with eigenvalue $-\big(s^2 + Q^2/4\big)$, as in [2, formula (2.6)]. The spherical Fourier transform $\mathcal{H}f$ of an integrable radial function f on S is defined by the formula

$$\mathcal{H}f(s) = \int_S \phi_s\, f \,\mathrm{d}\lambda\,.$$

For "nice" radial functions f on S, an inversion formula and a Plancherel formula hold:

$$f(x) = c_S \int_0^\infty \mathcal{H}f(s)\,\phi_s(x)\,|\mathbf{c}(s)|^{-2}\,\mathrm{d}s \qquad \forall x \in S\,,$$

and

$$\int_S |f|^2 \,\mathrm{d}\lambda = c_S \int_0^\infty |\mathcal{H}f(s)|^2\,|\mathbf{c}(s)|^{-2}\,\mathrm{d}s\,,$$

where the constant c_S depends only on m and k, and $\mathbf{c}$ denotes the Harish-Chandra function.

Let $\mathcal{A}$ denote the Abel transform and let $\mathcal{F}$ denote the Fourier transform on the real line, defined by

$$\mathcal{F}g(s) = \int_{-\infty}^{+\infty} g(r)\,\mathrm{e}^{-isr}\,\mathrm{d}r\,,$$

for each integrable function g on $\mathbb{R}$. It is well known that $\mathcal{H} = \mathcal{F}\circ\mathcal{A}$, hence $\mathcal{H}^{-1} = \mathcal{A}^{-1}\circ\mathcal{F}^{-1}$. For later use, we recall the inversion formula for the Abel transform [2, formula (2.24)]. We define the differential operators $\mathcal{D}_1$ and $\mathcal{D}_2$ on the real line by

$$\mathcal{D}_1 = -\frac{1}{\sinh r}\,\frac{\partial}{\partial r}\,, \qquad \mathcal{D}_2 = -\frac{1}{\sinh(r/2)}\,\frac{\partial}{\partial r}\,. \tag{7}$$

If k is even, then

$$\mathcal{A}^{-1}f(r) = a_S^e\, \mathcal{D}_1^{k/2}\mathcal{D}_2^{m/2} f(r)\,, \tag{8}$$

where $a_S^e = 2^{-(3m+k)/2}\pi^{-(m+k)/2}$, while if k is odd, then

$$\mathcal{A}^{-1}f(r) = a_S^o \int_r^\infty \mathcal{D}_1^{(k+1)/2}\mathcal{D}_2^{m/2} f(s)\,\mathrm{d}\nu(s)\,, \tag{9}$$

where $a_S^o = 2^{-(3m+k)/2}\pi^{-n/2}$ and $\mathrm{d}\nu(s) = (\cosh s - \cosh r)^{-1/2}\sinh s\,\mathrm{d}s$.

Let $\mathcal{L} = \sum_{i=0}^{n-1} X_i^2$ be the left invariant Laplacian defined in the Introduction. There is a special relationship between $\mathcal{L}$ and Δ_S. Indeed, denote by Δ_Q the shifted operator $\Delta_S + Q^2/4$; then by [5, Proposition 2],

$$\delta^{-1/2}(-\mathcal{L})\,\delta^{1/2} f = -\Delta_Q f \tag{10}$$

for all smooth compactly supported radial functions f on S. The spectra of $-\Delta_Q$ on $L^2(S,\lambda)$ and $-\mathcal{L}$ on $L^2(S,\rho)$ are both $[0,+\infty)$. Let E_{Δ_Q} and $E_{\mathcal{L}}$ be the spectral resolution of the identity for which

$$-\Delta_Q = \int_0^{+\infty} s\ \mathrm{d}E_{\Delta_Q}(s) \qquad \text{and} \qquad -\mathcal{L} = \int_0^{+\infty} s\ \mathrm{d}E_{\mathcal{L}}(s)\,.$$

For each bounded measurable function m on $\mathbb{R}^+$ the operators $m(-\Delta_Q)$ and $m(-\mathcal{L})$, spectrally defined by

$$m(-\Delta_Q) = \int_0^{+\infty} m(s)\,\mathrm{d}E_{\Delta_Q}(s) \qquad \text{and} \qquad m(-\mathcal{L}) = \int_0^{+\infty} m(s)\,\mathrm{d}E_{\mathcal{L}}(s)\,,$$

are bounded on $L^2(S,\lambda)$ and $L^2(S,\rho)$ respectively. By (10) and the spectral theorem,

$$\delta^{-1/2} m(-\mathcal{L})\,\delta^{1/2} f = m(-\Delta_Q) f\,, \tag{11}$$

for smooth compactly supported radial functions f on S. Let $k_{m(-\mathcal{L})}$ and $k_{m(-\Delta_Q)}$ denote the convolution kernels of $m(-\mathcal{L})$ and $m(-\Delta_Q)$ respectively; then

$$m(-\Delta_Q) f = f * k_{m(-\Delta_Q)} \qquad \text{and} \qquad m(-\mathcal{L}) f = f * k_{m(-\mathcal{L})} \qquad \forall f \in C_c^\infty(S)\,,$$

where $*$ denotes the convolution on S, defined by

$$f * g(x) = \int_S f(xy)\, g(y^{-1})\,\mathrm{d}\lambda(y) = \int_S f(xy^{-1})\, g(y)\,\mathrm{d}\rho(y)\,,$$

for all functions f, g in $C_c(S)$ and x in S. Given a bounded measurable function m on $\mathbb{R}^+$ the kernel $k_{m(-\Delta_Q)}$ is radial and

$$k_{m(-\mathcal{L})} = \delta^{1/2}\, k_{m(-\Delta_Q)}\,. \tag{12}$$

Moreover, the spherical transform $\mathcal{H}k_{m(-\Delta_Q)}$ of $k_{m(-\Delta_Q)}$ is given by

$$\mathcal{H}k_{m(-\Delta_Q)}(s) = m(s^2) \qquad \forall s \in \mathbb{R}^+\,. \tag{13}$$

For a proof of formula (13) see [2, 5].

3. Pointwise kernel estimates

We consider the general operator $e^{\tau\Delta_S}$, where $\tau = |\tau|\, e^{i\theta} \in \mathbb{C}\setminus\{0\}$ and denote by h_τ its convolution kernel. Our aim is to find pointwise estimates of this kernel when $\operatorname{Re}\tau \geq 0$. Notice that if $\tau \in \mathbb{R}^+$, then h_τ corresponds to the heat kernel and if $\tau = it \in i\mathbb{R}\setminus\{0\}$, then it corresponds to the Schrödinger kernel on Damek-Ricci spaces.

Notice that, for any $\tau \in \mathbb{C}\setminus\{0\}$ with $\operatorname{Re}\tau \geq 0$, we have $e^{\tau\Delta_S} = m_\tau(-\Delta_Q)$, where $m_\tau(v) = e^{-\frac{Q^2\tau}{4}-\tau v}$. Then by (13), the spherical Fourier transform of h_τ is

$$\mathcal{H}h_\tau(s) = m_\tau(s^2) = e^{-\frac{Q^2\tau}{4}}\, e^{-\tau s^2}\,,$$

and by applying the inverse Abel transform (8) and (9), we obtain the following formula for the kernel h_τ:

$$h_\tau(r) = \begin{cases} C\,(|\tau|e^{i\theta})^{-\frac12}\, e^{-\frac{Q^2\tau}{4}}\, \mathcal{D}_1^{k/2}\mathcal{D}_2^{m/2}\big(e^{-\frac{r^2}{4\tau}}\big) & \text{if } k \text{ even}\,,\\ C\,(|\tau|e^{i\theta})^{-\frac12}\, e^{-\frac{Q^2\tau}{4}}\, \int_r^\infty \mathcal{D}_1^{(k+1)/2}\mathcal{D}_2^{m/2}\big(e^{-\frac{s^2}{4\tau}}\big)\, \mathrm{d}\nu(s) & \text{if k odd}\,,\end{cases} \tag{14}$$

where $\mathcal{D}_1 = -\frac{1}{\sinh r}\frac{\partial}{\partial r}$ and $\mathcal{D}_2 = -\frac{1}{\sinh(r/2)}\frac{\partial}{\partial r}$. We now prove a pointwise estimate of the kernel h_τ.

Proposition 3.1. *There exists a positive constant C such that, for every $\tau \in \mathbb{C}^*$ with $\operatorname{Re}\tau \geq 0$ and for any $r \in \mathbb{R}^+$, we have*

$$|h_\tau(r)| \leq \begin{cases} C\,|\tau|^{-n/2}\,(1+r)^{\frac{n-1}{2}}\, e^{-\frac{Q}{2}r}\, e^{-\frac14\operatorname{Re}\{Q^2\tau+\frac{r^2}{\tau}\}} & \text{if } \ |\tau| \leq 1+r\,,\\ C\,|\tau|^{-3/2}\,(1+r)\, e^{-\frac{Q}{2}r}\, e^{-\frac14\operatorname{Re}\{Q^2\tau+\frac{r^2}{\tau}\}} & \text{if } \ |\tau| > 1+r\,.\end{cases} \tag{15}$$

Proof. We shall resume in part the analysis carried out in [2, Section 5] for the heat kernel, in [4, Proposition 3.1] and [26, 37] in the case of real hyperbolic spaces. Following the same ideas of [2, Corollary 5.21] by induction we can prove that for any $p,\, q \in \mathbb{N}$ such that $p + q \geq 1$

$$\mathcal{D}_1^q\, \mathcal{D}_2^p\big(e^{-\frac{r^2}{4\tau}}\big) = e^{-\frac{r^2}{4\tau}} \sum_{j=1}^{p+q} \tau^{-j}\, a_j(r)\,, \tag{16}$$

where a_j are finite linear combinations of products $f_{p_1,q_1}, \dots, f_{p_j,q_j}$ with $p_1 + \dots + p_j = p$, $q_1 + \dots + q_j = q$ and

$$f_{p,q}(r) \asymp (1+r)\, e^{-(p/2+q)r}.$$

Thus $a_j(r) = O\big((1+r)^j\, e^{-\frac{(p+2q)}{2}r}\big)$.

We first consider the case when k is even. By (14) and (16) we obtain that

$$\begin{aligned} |h_\tau(r)| &\lesssim |\tau|^{-1/2}\, e^{-\frac14\operatorname{Re}\{Q^2\tau+\frac{r^2}{\tau}\}} \sum_{j=1}^{(k+m)/2} |\tau|^{-j}\,(1+r)^j\, e^{-\frac{(m+2k)}{4}r}\\ &\lesssim |\tau|^{-1/2}\, e^{-\frac14\operatorname{Re}\{Q^2\tau+\frac{r^2}{\tau}\}}\, e^{-\frac{Q}{2}r}\left[\frac{1+r}{|\tau|} + \Big(\frac{1+r}{|\tau|}\Big)^{(n-1)/2}\right].\end{aligned}$$

This easily implies the desired estimate in this case.

Let now consider the case when k is odd. By (14) and (16) we obtain

$$|h_\tau(r)| \lesssim \sum_{j=1}^{(k+1+m)/2} |\tau|^{-j}\, |\tau|^{-1/2} \int_r^\infty \mathrm{d}s\, \frac{\sinh s}{\sqrt{\cosh s - \cosh r}}\, (1+s)^j \times \\ \times\, e^{-\frac{m+2(k+1)}{4}s}\, e^{-\frac14 \mathrm{Re}\,\{Q^2\tau + \frac{s^2}{\tau}\}}\,.$$

Here and throughout the proof, we make repeated use of the following elementary estimates:

$$\sinh s \asymp \frac{s}{1+s}\, e^s\,, \tag{17}$$

and

$$\cosh s - \cosh r \,=\, 2\,\sinh \frac{s-r}{2}\, \sinh \frac{s+r}{2} \asymp \frac{s-r}{1+s-r}\, \frac{s}{1+s}\, e^s \tag{18}$$

or

$$\cosh s - \cosh r \asymp \begin{cases} \frac{s^2-r^2}{1+r}\, e^r & \text{if } r \le s \le r+1\,, \\ \qquad e^s & \text{if } s \ge r+1\,. \end{cases} \tag{19}$$

By (17) and (18) we get

$$\begin{aligned} |h_\tau(r)| &\lesssim |\tau|^{-1/2}\, e^{-\frac14 \mathrm{Re}\,\{Q^2\tau\}} \int_r^\infty \mathrm{d}s \sqrt{\frac{1+s-r}{s-r}}\, \sqrt{\frac{1+s}{s}}\, e^{-s/2}\, \frac{s}{1+s}\, e^s \times \\ &\quad \times \Big[\frac{1+s}{|\tau|} + \Big(\frac{1+s}{|\tau|}\Big)^{n/2} \Big]\, e^{-\frac{Q}{2}s}\, e^{-\frac{s}{2}}\, e^{-\frac14 \mathrm{Re}\,\{\frac{s^2}{\tau}\}}. \end{aligned} \tag{20}$$

After performing the change of variables $s = r+u$, we obtain

$$|h_\tau(r)| \lesssim |\tau|^{-1/2}\, e^{-\frac14 \mathrm{Re}\,\{Q^2\tau\}} \int_0^\infty \mathrm{d}u \sqrt{\frac{1+u}{u}}\, \sqrt{\frac{r+u}{1+r+u}}\, e^{-\frac{Q}{2}(u+r)}\, e^{-\frac14 \mathrm{Re}\,\{\frac{u^2+r^2+2ur}{\tau}\}}.$$

Using the following inequalities

$$\frac{\sqrt{r+u}}{\sqrt{1+r+u}} \le 1\,, \quad 1+r+u \le (1+r)\,(1+u)\,,$$

we obtain

$$|\, h_\tau(r)\,| \,\lesssim\, |\tau|^{-\frac12}\, e^{-\frac{Q}{2}r}\, e^{-\frac14 \mathrm{Re}\,\{Q^2\tau + \frac{r^2}{\tau}\}} \Big\{ \frac{1+r}{|\tau|} + \Big(\frac{1+r}{|\tau|}\Big)^{\frac{n}{2}} \Big\}\,. \tag{21}$$

This allows us to obtain the desired estimate when $|\tau| > 1+r$.

If $|\tau| \le 1+r$, in order to prove the estimate (15), we need to reduce the power $\frac{n}{2}$ to $\frac{n-1}{2}$. For this purpose, inside (14), let us rewrite

$$\mathcal{D}_1^{(k+1)/2}\, \mathcal{D}_2^{m/2} (e^{-\frac{s^2}{4\tau}}) = P(\tau,s) + R(\tau,s)\,,$$

obtaining

$$h_\tau(r) = C\, (|\tau| e^{i\theta})^{-\frac12}\, e^{-\frac{Q^2\tau}{4}} \int_r^{+\infty} \mathrm{d}s\, \frac{\sinh s}{\sqrt{\cosh s \,-\, \cosh r}}\, \left[P(\tau,s) + R(\tau,s) \right], \tag{22}$$

where

$$P(\tau,s)=C\,\tau^{-\frac{(k+1+m)}{2}+1}\,s^{\frac{(k+1+m)}{2}-1}\Big(-\frac{1}{\sinh s}\Big)^{(k+1)/2}\Big(-\frac{1}{\sinh s/2}\Big)^{m/2}\frac{\partial}{\partial s}\big(e^{-\frac{s^2}{4\tau}}\big)$$

and $R(\tau,s)=\sum_{j=1}^{(k+1+m)/2-1}\tau^{-j}a_j(s)\,e^{-\frac{s^2}{4\tau}}$. Arguing as above, we can estimate the second term in the (22) as

$$\begin{aligned}&\Big||\tau|^{-\frac12}\,e^{-\frac{Q^2\tau}{4}}\int_r^\infty \mathrm{d}s\,\frac{\sinh s}{\sqrt{\cosh s\,-\,\cosh r}}R(\tau,s)\Big|\\ &\lesssim |\tau|^{-1/2}\,e^{-\frac14\mathrm{Re}\{Q^2\tau\}}\,e^{-\frac{Q}{2}r}\Big(\frac{1+r}{|\tau|}\Big)^{n/2-1}e^{-\frac14\mathrm{Re}\frac{r^2}{\tau}}\,.\end{aligned}\tag{23}$$

Hence, it remains to consider the integral

$$I(\tau,r)=\int_r^{+\infty}\mathrm{d}s\,\frac{\sinh s}{\sqrt{\cosh s\,-\,\cosh r}}\,P(\tau,s)\,,$$

when $|\tau|\le 1+r$. Let us write

$$I(\tau,r)\;=\;I_1(\tau,r)\;+\;I_2(\tau,r)\,,$$

according to the following splitting

$$\int_r^{+\infty}=\int_r^{\sqrt{r^2+|\tau|}}+\int_{\sqrt{r^2+|\tau|}}^{+\infty}.$$

To treat the first integral I_1, we differentiate $\frac{\partial}{\partial s}\big(e^{-\frac{s^2}{4\tau}}\big)=-\frac{s}{2\tau}\,e^{-\frac{s^2}{4\tau}}$ and use the estimates (17), (19) together with the fact that s is in $[\,r,r+1\,]$ obtaining

$$\begin{aligned}|I_1(\tau,r)|\;&\lesssim\;|\tau|^{-\frac{n}{2}}\,(1+r)^{\frac{n-1}{2}}\,e^{-\frac{Q}{2}r}\,e^{-\frac14\mathrm{Re}\{\frac{r^2}{\tau}\}}\int_r^{\sqrt{r^2+|\tau|}}\mathrm{d}s\,\frac{s}{\sqrt{s^2-r^2}}\\ &=\;|\tau|^{-\frac{n}{2}+\frac12}\,(1+r)^{\frac{n-1}{2}}\,e^{-\frac{Q}{2}r}\,e^{-\frac14\mathrm{Re}\{\frac{r^2}{\tau}\}}\,.\end{aligned}$$

By integrating by parts in the integral I_2, we get

$$I_2(\tau,r)=g(\tau,r)+J(\tau,r)\,,$$

where

$$\begin{aligned}&g(\tau,r)\\ &=\tau^{-\frac{n}{2}+1}\frac{\sinh s}{\sqrt{\cosh s\,-\,\cosh r}}\,s^{\frac{n}{2}-1}\Big(-\frac{1}{\sinh s}\Big)^{(k+1)/2}\Big(-\frac{1}{\sinh s/2}\Big)^{m/2}\big(e^{-\frac{s^2}{4\tau}}\big)\Big|_{s=\sqrt{r^2+|\tau|}}^{s=+\infty}\end{aligned}$$

and

$$\begin{aligned}J(\tau,r)=-\tau^{-\frac{n}{2}+1}&\int_{\sqrt{r^2+|\tau|}}^{+\infty}\mathrm{d}s\,e^{-\frac{s^2}{4\tau}}\times\\ &\times\frac{\partial}{\partial s}\Big[\frac{\sinh s}{\sqrt{\cosh s\,-\,\cosh r}}s^{\frac{n}{2}-1}\Big(-\frac{1}{\sinh s}\Big)^{(k+1)/2}\Big(-\frac{1}{\sinh s/2}\Big)^{m/2}\Big].\end{aligned}$$

We first estimate the boundary term $g(\tau,r)$, in the following way

$$\begin{aligned}|g(\tau,r)| &\lesssim |\tau|^{-\frac{n}{2}+1}\,(1+\sqrt{r^2+|\tau|})^{\frac{n}{2}-1}\,e^{-\frac{Q}{2}r}\,e^{-\frac14\mathrm{Re}\{\frac{r^2}{\tau}\}}\\ &\lesssim |\tau|^{-\frac{n}{2}+\frac12}\,(1+r)^{\frac{n-1}{2}}\,e^{-\frac{Q}{2}r}\,e^{-\frac14\mathrm{Re}\{\frac{r^2}{\tau}\}}\qquad \forall|\tau|\le 1+r.\end{aligned}$$

Then to estimate the integral term J we write it

$$J(\tau,r)=J_1(\tau,r)+J_2(\tau,r)\,,$$

according to

$$\int_{\sqrt{r^2+|\tau|}}^{+\infty}=\int_{\sqrt{r^2+|\tau|}}^{r+1}+\int_{r+1}^{+\infty}.$$

By computing the derivative which appears inside the first integral J_1 and using the elementary estimate $\frac{s\coth s-1}{\sinh s}\asymp s\,e^{-s}$, we obtain

$$\begin{aligned}|J_1(\tau,r)| &\lesssim |\tau|^{-\frac{n}{2}+1}\int_{\sqrt{r^2+|\tau|}}^{r+1}\mathrm{d}s\,(1+s)^{n/2-2}\,s\,e^{-\frac{Q}{2}s}\,e^{-\frac14\mathrm{Re}\{\frac{s^2}{\tau}\}}\Big\{e^{-\frac{r}{2}}\,e^{\frac{s}{2}}\Big(\frac{1+r}{s^2-r^2}\Big)^{1/2}\\ &\quad+e^{-\frac32 r}\,e^{\frac32 s}\,\Big(\frac{1+r}{s^2-r^2}\Big)^{3/2}\Big\}\\ &\lesssim |\tau|^{-\frac{n}{2}+1}\,e^{-\frac14\mathrm{Re}\{\frac{r^2}{\tau}\}}\,(1+r)^{\frac{n-3}{2}}\,e^{-\frac{Q}{2}r}\int_{\sqrt{r^2+|\tau|}}^{r+1}\mathrm{d}s\Big\{\frac{s}{\sqrt{s^2-r^2}}+(1+r)\,\frac{s}{(s^2-r^2)^{3/2}}\Big\}\\ &\lesssim |\tau|^{-\frac{n}{2}+\frac12}\,e^{-\frac14\mathrm{Re}\{\frac{r^2}{\tau}\}}\,e^{-\frac{Q}{2}r}\,(1+r)^{\frac{n-1}{2}}\qquad\forall|\tau|\le 1+r\,.\end{aligned}$$

We can estimate the second integral J_2 as above, changing variable $s=r+u$ and getting

$$\begin{aligned}|J_2(\tau,r)| &\lesssim |\tau|^{-\frac{n}{2}+1}\int_{r+1}^{\infty}\mathrm{d}s\,(1+s)^{n/2-2}\,s\,e^{-\frac{Q}{2}s}\,e^{-\frac14\mathrm{Re}\{\frac{s^2}{\tau}\}}\\ &\lesssim |\tau|^{-\frac{n}{2}+1}\,e^{-\frac14\mathrm{Re}\{\frac{r^2}{\tau}\}}\int_1^{\infty}\mathrm{d}u\,(1+r+u)^{n/2-2}\,(u+r)\,e^{-\frac{Q}{2}(u+r)}\\ &\lesssim |\tau|^{-\frac{n}{2}+1}\,e^{-\frac14\mathrm{Re}\{\frac{r^2}{\tau}\}}\,(1+r)^{n/2-1}\,e^{-\frac{Q}{2}r}\\ &\lesssim |\tau|^{-\frac{n}{2}+\frac12}\,e^{-\frac14\mathrm{Re}\{\frac{r^2}{\tau}\}}\,(1+r)^{\frac{n-1}{2}}\,e^{-\frac{Q}{2}r}\qquad\forall|\tau|\le 1+r\,.\end{aligned}$$

We have, thus, proved that

$$I(\tau,r)=O\big(|\tau|^{-\frac{n}{2}+\frac12}\,(1+r)^{\frac{n-1}{2}}\,e^{-\frac{Q}{2}r}\,e^{-\frac14\mathrm{Re}\{\frac{r^2}{\tau}\}}\big).$$

Then, the first term in (22) is estimated by

$$\big||\tau|^{-\frac12}\,e^{-\frac{Q^2}{4}\tau}\,I(\tau,r)\big|\lesssim|\tau|^{-\frac{n}{2}}\,(1+r)^{\frac{n-1}{2}}\,e^{-\frac{Q}{2}r}\,e^{-\frac14\mathrm{Re}\{\frac{r^2}{\tau}\}}\,,\tag{24}$$

which, combined with (23), allows to conclude the proof. □

Remark 3.2. *Notice that, in the case when $\tau\in\mathbb{R}^+$ estimates from below for the heat kernel h_τ were proved in* [26, Theorem 3.1] *for real hyperbolic spaces and* [2, Theorem 5.9] *for Damek-Ricci spaces. In Lemma 5.1, we shall prove an estimate from below for the kernel h_τ when $\tau\in i\mathbb{R}\setminus\{0\}$ in a suitable region of the space.*

As a consequence of the previous pointwise bounds of the kernel h_τ, we can estimate its norm in the weak Lorentz spaces.

Corollary 3.3. *Let* $2<q<\infty$ *and* $1\le\alpha\le\infty$ *. Then there exists a positive constant* C *such that the following kernel estimate holds*

$$\|h_\tau\|_{L^{q,\alpha}(S,\lambda)} \le \begin{cases} C\,|\tau|^{-n/2}\,e^{-\frac{Q^2}{4}\mathrm{Re}\tau} & \text{if } 0<|\tau|\le 1\,, \\ C\,|\tau|^{-3/2}\,e^{-\frac{Q^2}{4}\mathrm{Re}\tau} & \text{if } \quad |\tau|>1\,. \end{cases} \tag{25}$$

Proof. It suffices to argue as in [4, Lemma 3.2]. We recall that the Lorentz spaces $L^{q,\alpha}(S,\lambda)$ are the spaces of functions f such that :
if $1\le\alpha<\infty$, then

$$\|f\|_{L^{q,\alpha}(S,\lambda)} = \Big[\int_0^{+\infty}\frac{ds}{s}\,\{s^{1/q}f^*(s)\}^\alpha\Big]^{1/\alpha} = \Big[\int_0^{+\infty}dr\,\frac{V'(r)}{V(r)}\,\{V(r)^{1/q}f(r)\}^\alpha\Big]^{1/\alpha} < \infty,$$

or, if $\alpha=\infty$, then

$$\|f\|_{L^{q,\infty}(S,\lambda)} = \sup_{s>0} s^{1/q}f^*(s) = \sup_{r>0} V(r)^{1/q}f(r) < \infty,$$

where the decreasing function f^* is the rearrangement of f.
Notice that, if f is a radial decreasing function, then $f^*=f\circ V^{-1}$, where

$$\begin{aligned} V(r) &= C\int_0^r \mathrm{d}s\,\sinh^{m+k}(s/2)\,\cosh^k(s/2) \\ &\asymp \begin{cases} r^n & \text{as } r\to 0 \\ e^{Qr} & \text{as } r\to+\infty. \end{cases} \end{aligned}$$

Replacing f by h_τ and using the kernel estimate (15), we conclude the proof. □

4. Schrödinger equation on Damek-Ricci spaces

Consider the homogeneous Cauchy problem for the linear Schrödinger equation associated with the Laplace-Beltrami operator on a Damek-Ricci space S

$$\begin{cases} i\,\partial_t u(t,x) + \Delta_S u(t,x) = 0 \\ u(0,x) = f(x)\,, \quad x\in S\,, \end{cases} \tag{26}$$

whose solution is given by

$$u(t,x) = e^{it\Delta_S}f(x) = f*s_t(x)\,,$$

where we denote by s_t the kernel h_{it} for any $t\in\mathbb{R}\setminus\{0\}$. Our aim is to study the dispersive properties of $e^{it\Delta_S}$, and to do so we follow the strategy used in [4], where the authors applied the Kunze-Stein phenomenon. Notice that, for general functions on Damek-Ricci spaces, this phenomenon is known to be false [13, 17, 35, 36]. To solve this difficulty, we define suitable spaces of radial functions on S which have a “nice” convolution property.

Definition 4.1. *For s in $[2,\infty)$, we define $\mathcal{A}_s$ as the space of all radial function κ on S such that*
$$\int_0^\infty d\,|r\kappa(r)|^{s/2}\,\phi_0(r)\,A(r)<\infty\,,$$
where $A(r)$ is the radial density of the left measure, introduced in (6). *Given κ in $\mathcal{A}_s$, set*
$$\|\kappa\|_{\mathcal{A}_s}=\Big(\int_0^\infty dr\,|\kappa(r)|^{s/2}\,\phi_0(r)\,A(r)\Big)^{2/s}.$$

For $s=\infty$ we denote by $\mathcal{A}_\infty$ the space of $L^\infty(S,\lambda)$ radial functions on S and by $\|\cdot\|_{\mathcal{A}_\infty}$ the L^∞-norm.

We observe that $\mathcal{A}_s$ may be identified with the weighted space $L^{s/2}\big((0,\infty),\phi_0(r)\,A(r)\,\mathrm{d}r\big)$.

Theorem 4.2. *For any $q\in[2,\infty]$ we have that*
$$L^{q'}(S,\lambda)*\mathcal{A}_q\subset L^q(S,\lambda).$$
More precisely, there exists a constant C_q such that for every function f in $L^{q'}(S,\lambda)$ and κ in $\mathcal{A}_q$
$$\|f*\kappa\|_{L^q(S,\lambda)}\leq C_q\,\|\kappa\|_{\mathcal{A}_q}\,\|f\|_{L^{q'}(S,\lambda)}.$$

Proof. The case when $q=2$ follows by [2, Theorem (3.3)]. When $q=\infty$, taking f in $L^1(S,\lambda)$ and κ in $\mathcal{A}_\infty$, we have that for every x in S
$$|f*\kappa(x)|\leq\int_S|f(y)|\,|\kappa(y^{-1}x)|\,\mathrm{d}\lambda(y)\leq\|k\|_\infty\,\|f\|_{L^1(S,\lambda)}=\|k\|_{\mathcal{A}_{L^\infty}}\,\|f\|_{L^1(S,\lambda)}.$$
By interpolating between the case $q=2$ and $q=\infty$ we obtain that
$$[L^2(S,\lambda),L^1(S,\lambda)]_\theta\,*\,[\mathcal{A}_2,\mathcal{A}_\infty]_\theta\subset[L^2(S,\lambda),L^\infty(S,\lambda)]_\theta=L^q(S,\lambda)\tag{27}$$
where $1/q=(1-\theta)/2$, with $\theta\in(0,1)$ (see [8, Theorem 5.1.1]). Moreover, by [8, Theorem 5.1.1]
$$[L^2(S,\lambda),L^1(S,\lambda)]_\theta=L^{q'}(S,\lambda)$$
and
$$[L^1\big((0,\infty),\phi_0\,A\,\mathrm{d}r\big),L^\infty\big((0,\infty),\phi_0\,A\,\mathrm{d}r\big)]_\theta=L^{q/2}\big((0,\infty),\phi_0\,A\,\mathrm{d}r\big).$$
We then have $[\mathcal{A}_2,\mathcal{A}_\infty]_\theta=\mathcal{A}_q$, which combined with (27) implies the theorem. $\square$

Let us turn to $L^q\to L^{\tilde q'}$ dispersive properties of the propagator $e^{it\Delta_S}$ on S.

Theorem 4.3. *Let $2<q,\tilde q\leq\infty$. Then there exists a positive constant C such that, for all $t\in\mathbb{R}\setminus\{0\}$, the following dispersive estimates hold*
$$\|\,e^{it\Delta_S}\,\|_{L^{\tilde q'}(S,\lambda)\to L^q(S,\lambda)}\leq\begin{cases}C\,|t|^{-\max\{\frac12-\frac1q,\frac12-\frac1{\tilde q}\}\,n} & \text{if }\ 0<|t|\leq1\,,\\ C\,|t|^{-\frac32} & \text{if }\ \ |t|>1\,.\end{cases}$$

Proof. For $0 < |t| \leq 1$, by applying Corollary 3.3, we obtain

$$\begin{cases} \| \, e^{it\Delta_S} \, \|_{L^1(S,\lambda)\to L^q(S,\lambda)} = \|s_t\|_{L^q(S,\lambda)} \leq C\, |t|^{-\frac{n}{2}} \quad \forall\, q>2\,, \\ \| \, e^{it\Delta_S} \, \|_{L^{q'}(S,\lambda)\to L^\infty(S,\lambda)} = \|s_t\|_{L^q(S,\lambda)} \leq C\, |t|^{-\frac{n}{2}} \quad \forall\, q>2\,, \\ \| \, e^{it\Delta_S} \, \|_{L^2(S,\lambda)\to L^2(S,\lambda)} = 1\,. \end{cases}$$

By interpolating the previous estimates, we deduce the desired result.

When $|t| > 1$, we apply Corollary 3.3 and Theorem 4.2 obtaining

$$\begin{cases} \| \, e^{it\Delta_S} \, \|_{L^1(S,\lambda)\to L^q(S,\lambda)} = \|s_t\|_{L^q(S,\lambda)} \leq C\, |t|^{-\frac{3}{2}} \quad \forall\, q>2\,, \\ \| \, e^{it\Delta_S} \, \|_{L^{q'}(S,\lambda)\to L^\infty(S,\lambda)} = \|s_t\|_{L^q(S,\lambda)} \leq C\, |t|^{-\frac{3}{2}} \quad \forall\, q>2\,, \\ \| \, e^{it\Delta_S} \, \|_{L^{q'}(S,\lambda)\to L^q(S,\lambda)} \leq C_q\, \|s_t\|_{\mathcal{A}_q} \quad \forall\, q>2\,. \end{cases} \tag{28}$$

For any $2 < q \leq \infty$, we have to estimate the $\mathcal{A}_q$-norm of the kernel s_t. To do so, we use Proposition 3.1, formula (6) and the following inequality (see [5, Lemma 1])

$$\phi_0(r) \leq C\,(1+r)\, e^{-\frac{Q}{2}r}\,,$$

from which we deduce that s_t lies in the space $\mathcal{A}_q$ and

$$\|s_t\|_{\mathcal{A}_q} \lesssim \; |t|^{-3/2} \qquad \forall |t| > 1\,.$$

Using the previous estimate in (28) and applying interpolation, we conclude the proof when $|t|$ is large. □

Finally, by combining dispersive estimates (proved in Theorem 4.3) with the classical TT^* method in the same way of [4, Theorem 3.6], we deduce the Strichartz estimates for a large family of admissible pairs.

Theorem 4.4. *Consider the Cauchy Problem for the linear Schrödinger equation*

$$\begin{cases} i\partial_t u(t,x) + \Delta_S u(t,x) = F(t,x) \\ u(0,x) = f(x), \;\; x \in S. \end{cases}$$

If $(\frac{1}{p}, \frac{1}{q})$ *and* $(\frac{1}{\tilde{p}}, \frac{1}{\tilde{q}})$ *lie in the admissible triangle*

$$T_n = \Big\{ \Big(\frac{1}{p}, \frac{1}{q}\Big) \in \Big(0, \frac{1}{2}\Big] \times \Big(0, \frac{1}{2}\Big) : \; \frac{2}{p} + \frac{n}{q} \geq \frac{n}{2} \Big\} \cup \Big\{ \Big(0, \frac{1}{2}\Big) \Big\}, \tag{29}$$

then the solution

$$u(t,x) = e^{it\Delta_S} f(x) + \int_0^t ds\, e^{i(t-s)\Delta_S} F(s,x)$$

satisfies the following Strichartz estimates

$$\|u\|_{L^p(\mathbb{R};L^q(S,\lambda))} \lesssim \|f\|_{L^2(S,\lambda)} + \|F\|_{L^{\tilde{p}'}(\mathbb{R};L^{\tilde{q}'}(S,\lambda))}\,.$$

Remark 4.5. *The applications to well-posedness and scattering theory for the NLS obtained in* [4, Section 4, 5] *on real hyperbolic spaces, can be easily generalized to all Damek-Ricci spaces. We omit the details.*

5. An application: the Schrödinger equation associated with $\mathcal{L}$

We now consider the homogeneous Cauchy problem for the linear Schrödinger equation on a Damek-Ricci space S associated with the distinguished Laplacian $\mathcal{L}$

$$\begin{cases} i\partial_t u(t,x) + \mathcal{L}u(t,x) = 0 \\ u(0,x) = f(x), \ \ x \in S, \end{cases}$$

whose solution is given by

$$u(t,x) = f * \sigma_t(x)\,,$$

where σ_t is the convolution kernel of the operator $e^{it\mathcal{L}}$.

It is interesting to observe that $L^1 - L^\infty$ dispersive estimate for the Schrödinger equation associated with the Laplacian $\mathcal{L}$ does not hold. To show this fact, we first prove that the kernel σ_t is not in L^∞. More precisely, we estimate from below the kernel s_t and we use the relationship (12) between kernels of multipliers of $-\Delta_S$ and $-\mathcal{L}$,

$$\sigma_t = \delta^{1/2}\, e^{i\frac{Q^2 t}{4}}\, s_t, \tag{30}$$

where δ is the modular function on S.

Lemma 5.1. *For every t in $\mathbb{R}\setminus\{0\}$ there exist positive constants K and c, with $c>1$, such that*

$$|s_t(r)| \geq K\,|t|^{-n/2}\, r^{\frac{n-1}{2}}\, e^{-\frac{Q}{2}r} \qquad \forall r > 1 + c|t|.$$

Proof. We suppose for simplicity $t>0$. We first consider the case when k is even. By the expression (14) of the kernel and the expansion (16), we obtain

$$\begin{aligned} s_t(r) &= C\,t^{-1/2}\, e^{i\frac{Q^2 t}{4}}\, e^{i\frac{r^2}{4t}} \sum_{j=1}^{(k+m)/2-1} t^{-j}\, a_j(r) \\ &\quad + C\,t^{-1/2}\, t^{-(k+m)/2+1}\, r^{(k+m)/2-1} \Big(-\frac{1}{\sinh r}\Big)^{k/2-1} \Big(-\frac{1}{\sinh(r/2)}\Big)^{m/2} \mathcal{D}_1(e^{i\frac{r^2}{4t}}) \\ &\quad + C\,t^{-1/2}\, t^{-(k+m)/2+1}\, r^{(k+m)/2-1} \Big(-\frac{1}{\sinh r}\Big)^{k/2} \Big(-\frac{1}{\sinh(r/2)}\Big)^{m/2-1} \mathcal{D}_2(e^{i\frac{r^2}{4t}}) \\ &= A(t,r) + B(t,r)\,, \end{aligned} \tag{31}$$

where $A(t,r)$ corresponds to the sum and $B(t,r)$ comes from the last two summands. By computing the derivatives which appear in the term $B(t,r)$, we obtain

$$B(t,r) = C\,t^{-1/2}\, t^{-(n-1)/2} \Big(\frac{ir}{2}\Big)^{(n-1)/2} \Big(-\frac{1}{\sinh r}\Big)^{k/2} \Big(-\frac{1}{\sinh(r/2)}\Big)^{m/2} e^{i\frac{r^2}{4t}}.$$

As in the proof of Proposition 3.1, we see that

$$A(t,r) = O(t^{-n/2+1}\, r^{(n-1)/2-1}\, e^{-\frac{Q}{2}r}) \qquad \forall r > 1+t\,, \tag{32}$$

and there exists a positive constant C such that

$$|B(t,r)| \geq C\,t^{-n/2}\, r^{(n-1)/2}\, e^{-\frac{Q}{2}r} \qquad \forall r > 1+t\,. \tag{33}$$

By (32) and (33), we deduce that there exists a sufficently large constant c and a positive constant K such that $|s_t(r)| \geq K\, t^{-n/2}\, r^{(n-1)/2}\, e^{-\frac{Q}{2}r}$, $\forall r > 1 + c\,t$, as required.

Suppose now that k is odd. As before, by (14) and (16), we can write

$$s_t(r) = \tilde{A}(t,r) + \tilde{B}(t,r)\,, \tag{34}$$

where $\tilde{A}(t,r) = C\,t^{-1/2}\sum_{j=1}^{(k+1+m)/2-1}\int_r^\infty t^{-j}\, a_j(s)\, e^{i\frac{s^2}{4t}}\,\mathrm{d}\nu(s)$ and

$$\tilde{B}(t,r) = C\,t^{-1/2}\,t^{-n/2+1}\int_r^\infty s^{n/2-1}\Big(\frac{1}{\sinh s}\Big)^{(k+1)/2}\Big(\frac{1}{\sinh(s/2)}\Big)^{m/2}\frac{\partial}{\partial s}\Big(e^{i\frac{s^2}{4t}}\Big)\,\mathrm{d}\nu(s)\,. \tag{35}$$

As in the proof of Proposition 3.1, we see that

$$\tilde{A}(t,r) = O(t^{-(n-1)/2}\, r^{n/2-1}\, e^{-\frac{Q}{2}r})\,. \tag{36}$$

Since

$$\mathrm{d}\nu(s) = \frac{\sinh s}{\sqrt{\cosh s - \cosh r}} = \sinh s\,\Big(2\,\sinh\frac{s+r}{2}\,\sinh\frac{s-r}{2}\Big)^{-1/2},$$

the main term $\tilde{B}(t,r)$ can be written as

$$\begin{aligned}
\tilde{B}(t,r) &= C\,t^{-(n+1)/2}\, e^{i\frac{r^2}{4t}}\int_r^\infty \mathrm{d}s\, s\,\Big(\sinh\frac{s+r}{2}\,\sinh\frac{s-r}{2}\Big)^{-1/2}\Big(\frac{s}{\sinh s}\Big)^{(k-1)/2}\times \\
&\quad\times\Big(\frac{s}{\sinh s/2}\Big)^{m/2} e^{i\frac{s^2-r^2}{4t}} \\
&= C\,t^{-(n+1)/2}\, e^{i\frac{r^2}{4t}}\int_r^\infty \mathrm{d}s\, s\, f_r(s)\, e^{i\frac{s^2-r^2}{4t}}\,,
\end{aligned}$$

where $f_r(s) = \Big(\sinh\frac{s+r}{2}\,\sinh\frac{s-r}{2}\Big)^{-1/2}\Big(\frac{s}{\sinh s}\Big)^{(k-1)/2}\Big(\frac{s}{\sinh s/2}\Big)^{m/2}$. By changing variables $u = \frac{s^2-r^2}{4t}$ the integral transforms into

$$\tilde{B}(t,r) = C\,t^{-\frac{n-1}{2}}\, e^{\frac{ir^2}{4t}}\int_0^{+\infty}\mathrm{d}u\, e^{iu}\, f_r(s(u)).$$

Hence

$$|\tilde{B}(t,r)| \geq |C|\,t^{-\frac{n-1}{2}}\,\mathrm{Im}\Big\{\int_0^{+\infty}\mathrm{d}u\, e^{iu}\, f_r(s(u))\Big\}\,,$$

which can be split up in the following sum

$$|C|\,t^{-\frac{n-1}{2}}\sum_{j=0}^{+\infty}\int_{2j\pi}^{(2j+1)\pi}\mathrm{d}u\,\sin u\,\big\{f_r(s(u)) - f_r(s(u+\pi))\big\}\,,$$

which, since $u \longmapsto f_r(s(u))$ is a positive decreasing function, is estimated from below by

$$|C|\,t^{-\frac{n-1}{2}}\int_0^\pi \mathrm{d}u\,\sin u\,\big\{f_r(s(u)) - f_r(s(u+\pi))\big\}\,.$$

To estimate the last integral we write

$$f_r(s(u))-f_r(s(u+\pi))=\int_0^{\pi}\mathrm{d}v\,\{-f_r'(s(u{+}v)\}\,s'(u{+}v)\,.$$

Notice that $s(u)=\sqrt{4tu+r^2}$, so that $s'(u)=\frac{2t}{s(u)}$. We now compute the derivative of $-f_r$ obtaining

(37)

$$\begin{aligned}-f_r'(s)=&\ \frac14\Big(\sinh\frac{s+r}{2}\sinh\frac{s-r}{2}\Big)^{-\frac32}\sinh s\,\Big(\frac{s}{\sinh s}\Big)^{(k-1)/2}\Big(\frac{s}{\sinh s/2}\Big)^{m/2}\\ &+\Big(\sinh\frac{s+r}{2}\sinh\frac{s-r}{2}\Big)^{-\frac12}\Big[\frac{k-1}{2}\Big(\frac{s}{\sinh s}\Big)^{(k-3)/2}\frac{s\coth s-1}{\sinh s}\Big(\frac{s}{\sinh s/2}\Big)^{m/2}\\ &+\Big(\frac{s}{\sinh s}\Big)^{(k-1)/2}\frac{m}{2}\Big(\frac{s}{\sinh s/2}\Big)^{m/2-1}\frac{\frac{s}{2}\coth(\frac{s}{2})-1}{\sinh\frac{s}{2}}\Big]\,.\end{aligned}$$

We now use in (37) the elementary estimates

$$\sinh s\asymp e^s,\qquad \sinh(s/2)\asymp e^{s/2},\qquad s\coth s-1\asymp s,$$

and

$$\sinh\frac{s+r}{2}\sinh\frac{s-r}{2}\asymp\frac{s^2-r^2}{s}\,e^s\,,$$

to obtain

$$-f_r'(s)\asymp(s^2-r^2)^{-1/2}\,s^{\frac{n-1}{2}}\,e^{-\frac{Q}{2}s}\,\big[(s^2-r^2)^{-1}\,s+2\big]\,.$$

By replacing $s=s(u+v)=\sqrt{4t(u+v)+r^2}$, we get

$$-f_r'\big(s(u{+}v)\big)\asymp\big(4t(u{+}v)\big)^{-1/2}\,s(u{+}v)^{\frac{n-1}{2}}\,e^{-\frac{Q}{2}s(u+v)}\,\big[\big(4t(u{+}v)\big)^{-1}\,s(u{+}v)+2\big]\,. \tag{38}$$

Observe that, in the integral defining $\tilde{B}(t,r)$, we have $1\le r\le s$ and

$$s=r\sqrt{1{+}\,4\,(u{+}v)\,\frac{t}{r^2}}\,\,.$$

Since $0<u+v<2\pi$ and $r>1+t$, we deduce

$$s(u+v)\lesssim r\Big[1+\frac{4(u+v)\frac{t}{r^2}}{2}\Big]\lesssim r+2(u+v)\frac{t}{r}\lesssim r+4\,\pi\,.$$

From (38) and the previous estimates, we get

$$\begin{aligned}-f_r'(s(u{+}v))\,s'(u{+}v)&\asymp\big(4t(u+v)\big)^{-1/2}\,r^{\frac{n-1}{2}}\,e^{-\frac{Q}{2}r}\,\big[\big(4t(u+v)\big)^{-1}\,r+2\big]\,\frac{2t}{r}\\ &\asymp\big(4t(u+v)\big)^{-1/2}\,r^{\frac{n-1}{2}}\,e^{-\frac{Q}{2}r}\,\big[\big(2(u+v)\big)^{-1}+\frac{4t}{r}\big]\\ &\asymp\,(u{+}v)^{-\frac32}\,t^{-\frac12}\,r^{\frac{n-1}{2}}\,e^{-\frac{Q}{2}r}\qquad\forall r>1+t\,.\end{aligned}\tag{39}$$

Hence

$$f_r(s(u))-f_r(s(u+\pi))\asymp\,u^{-\frac12}\,t^{-\frac12}\,r^{\frac{n-1}{2}}\,e^{-\frac{Q}{2}r},$$

so that we obtain

$$(40)\quad |\tilde{B}(t,r)| \geq |C|\, t^{-\frac{n-1}{2}} \int_0^\pi \mathrm{d}u\, \sin u \,\big\{\, f_r(s(u)) - f_r(s(u+\pi)) \,\big\} \geq C\, t^{-\frac{n}{2}}\, r^{\frac{n-1}{2}}\, e^{-\frac{Q}{2}r}.$$

By (36) and (40), we see that there exists a sufficiently large constant c and a positive constant K such that $|s_t(r)| \geq K\, t^{-\frac{n}{2}}\, r^{\frac{n-1}{2}}\, e^{-\frac{Q}{2}r}$ for all $r > 1 + ct$, as required. □

Proposition 5.2. *For every t in $\mathbb{R} \setminus \{0\}$, the following hold:*
- (i) *the kernel σ_t does not lie in $L^\infty(S,\rho)$;*
- (ii) *the operator $e^{it\mathcal{L}}$ is not bounded from $L^1(S,\rho)$ to $L^\infty(S,\rho)$.*

Proof. Since $\sigma_t = \delta^{1/2}\, e^{i\frac{Q^2 t}{4}}\, s_t$, from Lemma 5.1 we deduce that there exist constants $c > 1$ and $K > 0$ for which

$$|\sigma_t(x)| \geq K\, |t|^{-n/2}\, \delta^{1/2}(x)\, r(x)^{\frac{n-1}{2}}\, e^{-\frac{Q}{2}r(x)} \qquad \forall r(x) > 1 + c|t|\,.$$

Let Ω_t be the following region:

$$\Omega_t = \{x = (X,Z,a) \in \mathfrak{v} \times \mathfrak{z} \times \mathbb{R}^+ :\ r(x) > 1 + c|t|, a < 1, |(X,Z)| < 1\}.$$

By formula (5), for any point (X,Z,a) in Ω_t, we have

$$e^{r(X,Z,a)} \asymp a^{-1} \qquad \text{and} \qquad r(X,Z,a) \asymp \log(a^{-1}).$$

Hence for any point (X,Z,a) in Ω_t

$$|\sigma_t(X,Z,a)| \geq C\, a^{-Q/2}\, |t|^{-n/2}\, [\log(a^{-1})]^{\frac{n-1}{2}}\, a^{Q/2} \geq C\, |t|^{-n/2}\, [\log(a^{-1})]^{\frac{n-1}{2}}.$$

This shows that σ_t is not in $L^\infty(S,\rho)$ and proves (i).

Let now ϕ_n be a sequence of approximations of the identity, i.e. functions in $C_c^\infty(S)$ supported in the ball centred at the identity of radius $1/n$ such that $\|\phi_n\|_{L^1(S,\rho)} = 1$, $0 \leq \phi_n \leq 1$. Suppose that the operator $e^{it\mathcal{L}}$ is bounded from $L^1(S,\rho)$ to $L^\infty(S,\rho)$. Then there exists a constant M such that $\|\phi_n * \sigma_t\|_{L^\infty(S,\rho)} \leq M$. Since $\phi_n * \sigma_t$ converges to σ_t almost everywhere we deduce that $|\sigma_t| \leq M$ almost everywhere which contradicts (i). Thus the operator $e^{it\mathcal{L}}$ is not bounded from $L^1(S,\rho)$ to $L^\infty(S,\rho)$. □

Even if the $L^1 - L^\infty$ dispersive estimate does not hold, we shall prove suitable weighted Strichartz estimates for the Schrödinger equation associated with the Laplacian $\mathcal{L}$. We shall deduce them from the Strichartz estimates which hold for the Schrödinger equation associated with the Laplace-Beltrami operator. To do so, for any $q \in [2,\infty)$ we introduce the weight function δ_q defined by

$$(41)\qquad \delta_q = \delta^{1-q/2} = \delta^{q\left(\frac{1}{q}-\frac{1}{2}\right)}.$$

The weights δ_q are involved in a simple relationship between the L^q norms of functions computed with respect to the right and left Haar measures.

Lemma 5.3. *For any $q \in [2,\infty)$, the following hold:*
- (i) $\|\delta^{-1/2} f\|_{L^q(S,\lambda)} = \|f\|_{L^q(S,\delta_q\rho)}$ *for every f in $L^q(S,\delta_q\rho)$;*
- (ii) $\|f\|_{L^q(S,\lambda)} = \|\delta^{1/2} f\|_{L^q(S,\delta_q\rho)}$ *for every f in $L^q(S,\lambda)$.*

Proof. Take f in $L^q(S, \delta_q \rho)$. We have that

$$
\begin{aligned}
\|\delta^{-1/2} f\|^q_{L^q(S,\lambda)} &= \int \delta^{-q/2}\, |f|^q \,\mathrm{d}\lambda \\
&= \int \delta^{-q/2}\, |f|^q \,\delta\,\mathrm{d}\rho \\
&= \int \delta^{q(1/q-1/2)}\, |f|^q \,\mathrm{d}\rho \\
&= \|f\|^q_{L^q(S,\delta_q\rho)}.
\end{aligned}
\tag{42}
$$

This proves (i). The statement (ii) follows directly from (i). □

Theorem 5.4. *Consider the Cauchy Problem for the linear Schrödinger equation*

$$
\begin{cases}
i\partial_t u(t,x) + \mathcal{L}u(t,x) = F(t,x) \\
u(0,x) = f(x), \;\; x \in S.
\end{cases}
$$

For all $\left(\frac{1}{p}, \frac{1}{q}\right)$ *and* $\left(\frac{1}{\tilde{p}}, \frac{1}{\tilde{q}}\right)$ *in the admissible triangle* T_n*, the solution*

$$
u(t,x) = e^{it\mathcal{L}} f(x) + \int_0^t ds\, e^{i(t-s)\mathcal{L}} F(s,x), \tag{43}
$$

satisfies the following weighted Strichartz estimates

$$
\|u\|_{L^p(\mathbb{R};\, L^q(S,\delta_q\,\rho))} \lesssim \|f\|_{L^2(S,\rho)} + \|F\|_{L^{\tilde{p}'}(\mathbb{R};\, L^{\tilde{q}'}(S,\delta_{\tilde{q}'}\,\rho))}\,.
$$

Proof. By (30), we deduce that

$$
u(t,x) = e^{i\frac{Q^2 t}{4}}\, f * (\delta^{1/2}\, s_t)(x) + \int_0^t \mathrm{d}s\, e^{i\frac{Q^2(t-s)}{4}} \left[F * (\delta^{1/2}\, s_{(t-s)})\right](s,x). \tag{44}
$$

It is easy to see that for any functions h, g on S

$$
h * (\delta^{1/2} g) = \delta^{1/2}\left[\left(\delta^{-1/2} h\right) * g\right]. \tag{45}
$$

Applying (45) in (44), we obtain

$$
e^{-i\frac{Q^2 t}{4}}\, \delta^{-1/2}\, u(t,x) = \left(\delta^{-1/2} f\right) * s_t(x) + \int_0^t \mathrm{d}s\, e^{-i\frac{Q^2 s}{4}} \left[\delta^{-1/2} F * s_{(t-s)}\right](s,x)\,. \tag{46}
$$

Suppose now that $\left(\frac{1}{p}, \frac{1}{q}\right)$ and $\left(\frac{1}{\tilde{p}}, \frac{1}{\tilde{q}}\right)$ lie in the admissible triangle T_n introduced in (29). By Lemma 5.3 and Theorem 4.4, we get

$$
\begin{aligned}
\|u\|_{L^p(\mathbb{R}; L^q(S,\delta_q\,\rho))} &= \|\delta^{-1/2} u\|_{L^p(\mathbb{R}; L^q(S,\lambda))} \\
&\lesssim \|\delta^{-1/2} f\|_{L^2(S,\lambda)} + \|\delta^{-1/2} F\|_{L^{\tilde{p}'}(\mathbb{R}; L^{\tilde{q}'}(S,\lambda))} \\
&= \|f\|_{L^2(S,\rho)} + \|F\|_{L^{\tilde{p}'}(\mathbb{R}; L^{\tilde{q}'}(S,\delta_{\tilde{q}'}\,\rho))},
\end{aligned}
\tag{47}
$$

as required. □

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